医学部予備校　慧修会

東海大学医学部　最後の一題

問題

複素数 $\alpha=\cos\dfrac{2\pi}{7}+i\sin\dfrac{2\pi}{7}$ に対し，次の式の値を求めよ。ただし，i は虚数単位とする。

(1) $\alpha+\alpha^2+\alpha^3+\alpha^4+\alpha^5+\alpha^6$

(2) $\dfrac{1}{1-\alpha}+\dfrac{1}{1-\alpha^6}$

(3) $\dfrac{1}{1-\alpha}+\dfrac{1}{1-\alpha^2}+\dfrac{1}{1-\alpha^3}+\dfrac{1}{1-\alpha^4}+\dfrac{1}{1-\alpha^5}+\dfrac{1}{1-\alpha^6}$

(4) $\dfrac{\alpha^2}{1-\alpha}+\dfrac{\alpha^4}{1-\alpha^2}+\dfrac{\alpha^6}{1-\alpha^3}+\dfrac{\alpha^8}{1-\alpha^4}+\dfrac{\alpha^{10}}{1-\alpha^5}+\dfrac{\alpha^{12}}{1-\alpha^6}$

最後の確認

(1)
$$\alpha^7=\cos 2\pi+i\sin 2\pi=1$$
より
$$\alpha^7-1=(\alpha-1)(\alpha^6+\alpha^5+\alpha^4+\alpha^3+\alpha^2+\alpha+1)=0$$
$\alpha\fallingdotseq 1$ であるから
$$\alpha^6+\alpha^5+\alpha^4+\alpha^3+\alpha^2+\alpha=-1$$

(2)
$$(\text{与式})=\frac{1}{1-\alpha}+\frac{\alpha}{\alpha-\alpha^7}=\frac{1}{1-\alpha}+\frac{\alpha}{\alpha-1}=\frac{1-\alpha}{1-\alpha}=1$$

(3)
$$(\text{与式})=\left(\frac{1}{1-\alpha}+\frac{1}{1-\alpha^6}\right)+\left(\frac{1}{1-\alpha^2}+\frac{1}{1-\alpha^5}\right)+\left(\frac{1}{1-\alpha^3}+\frac{1}{1-\alpha^4}\right)$$
ここで，(2) と同様に
$$\frac{1}{1-\alpha^2}+\frac{1}{1-\alpha^5}=\frac{1}{1-\alpha^2}+\frac{\alpha^2}{\alpha^2-1}=1,$$
$$\frac{1}{1-\alpha^3}+\frac{1}{1-\alpha^4}=\frac{1}{1-\alpha^3}+\frac{\alpha^3}{\alpha^3-1}=1$$
したがって
$$(\text{与式})=1+1+1=3$$

(4) (与式)$=\left(-\alpha-1+\dfrac{1}{1-\alpha}\right)+\left(-\alpha^2-1+\dfrac{1}{1-\alpha^2}\right)+\left(-\alpha^3-1+\dfrac{1}{1-\alpha^3}\right)$

$+\left(-\alpha^4-1+\dfrac{1}{1-\alpha^4}\right)+\left(-\alpha^5-1+\dfrac{1}{1-\alpha^5}\right)+\left(-\alpha^6-1+\dfrac{1}{1-\alpha^6}\right)$

$=-6-(\alpha+\alpha^2+\alpha^3+\alpha^4+\alpha^5+\alpha^6)+\dfrac{1}{1-\alpha}+\dfrac{1}{1-\alpha^2}+\dfrac{1}{1-\alpha^3}+\dfrac{1}{1-\alpha^4}$

$+\dfrac{1}{1-\alpha^5}+\dfrac{1}{1-\alpha^6}$

$=-6-(-1)+3=-2$