

東海大学医学部 最後の一題

問題

複素数 $\alpha = \cos \frac{2\pi}{7} + i \sin \frac{2\pi}{7}$ に対し、次の式の値を求めよ。ただし、 i は虚数単位とする。

(1) $\alpha + \alpha^2 + \alpha^3 + \alpha^4 + \alpha^5 + \alpha^6$

(2) $\frac{1}{1-\alpha} + \frac{1}{1-\alpha^6}$

(3) $\frac{1}{1-\alpha} + \frac{1}{1-\alpha^2} + \frac{1}{1-\alpha^3} + \frac{1}{1-\alpha^4} + \frac{1}{1-\alpha^5} + \frac{1}{1-\alpha^6}$

(4) $\frac{\alpha^2}{1-\alpha} + \frac{\alpha^4}{1-\alpha^2} + \frac{\alpha^6}{1-\alpha^3} + \frac{\alpha^8}{1-\alpha^4} + \frac{\alpha^{10}}{1-\alpha^5} + \frac{\alpha^{12}}{1-\alpha^6}$

最後の確認

(1) $\alpha^7 = \cos 2\pi + i \sin 2\pi = 1$ より

$$\alpha^7 - 1 = (\alpha - 1)(\alpha^6 + \alpha^5 + \alpha^4 + \alpha^3 + \alpha^2 + \alpha + 1) = 0$$

$\alpha \neq 1$ であるから

$$\alpha^6 + \alpha^5 + \alpha^4 + \alpha^3 + \alpha^2 + \alpha = -1$$

別解

$\alpha \neq 1$ であるから、等比数列の和の公式により

$$\alpha + \alpha^2 + \alpha^3 + \alpha^4 + \alpha^5 + \alpha^6 = \frac{\alpha(1-\alpha^6)}{1-\alpha} = \frac{\alpha-\alpha^7}{1-\alpha} = \frac{\alpha-1}{1-\alpha} = -1$$

(2) (与式) $= \frac{1}{1-\alpha} + \frac{\alpha}{\alpha-\alpha^7} = \frac{1}{1-\alpha} + \frac{\alpha}{\alpha-1} = \frac{1-\alpha}{1-\alpha} = 1$

(3) (与式) $= \left(\frac{1}{1-\alpha} + \frac{1}{1-\alpha^6} \right) + \left(\frac{1}{1-\alpha^2} + \frac{1}{1-\alpha^5} \right) + \left(\frac{1}{1-\alpha^3} + \frac{1}{1-\alpha^4} \right)$

ここで、(2)と同様に

$$\frac{1}{1-\alpha^2} + \frac{1}{1-\alpha^5} = \frac{1}{1-\alpha^2} + \frac{\alpha^2}{\alpha^2-1} = 1,$$

$$\frac{1}{1-\alpha^3} + \frac{1}{1-\alpha^4} = \frac{1}{1-\alpha^3} + \frac{\alpha^3}{\alpha^3-1} = 1$$

したがって

$$(与式) = 1 + 1 + 1 = 3$$

$$\begin{aligned} (4) \text{ (与式)} &= \left(-\alpha - 1 + \frac{1}{1-\alpha}\right) + \left(-\alpha^2 - 1 + \frac{1}{1-\alpha^2}\right) + \left(-\alpha^3 - 1 + \frac{1}{1-\alpha^3}\right) \\ &\quad + \left(-\alpha^4 - 1 + \frac{1}{1-\alpha^4}\right) + \left(-\alpha^5 - 1 + \frac{1}{1-\alpha^5}\right) + \left(-\alpha^6 - 1 + \frac{1}{1-\alpha^6}\right) \\ &= -6 - (\alpha + \alpha^2 + \alpha^3 + \alpha^4 + \alpha^5 + \alpha^6) + \frac{1}{1-\alpha} + \frac{1}{1-\alpha^2} + \frac{1}{1-\alpha^3} + \frac{1}{1-\alpha^4} \\ &\quad + \frac{1}{1-\alpha^5} + \frac{1}{1-\alpha^6} \\ &= -6 - (-1) + 3 = -2 \end{aligned}$$